Registration No.: | | | | | | | | | | |

Total Number of Pages: 02

Course: M.Sc.I
Sub_Code: FMCC702

7th Semester Regular Examination: 2024-25
SUBJECT: Measure Theory
BRANCH(S): M.Sc.I(MC)
Time: 3 Hours
Max Marks: 70
Q.Code: R087
Answer Question No.1 (Part-I) which is compulsory, any five from rest (Part-II)
The figures in the right hand margin indicate marks.

Part-I

Q1 Answer the following questions: (2 x 10)

- **a)** Define perfect set. Give an example of it.
- **b)** Write two properties of Lebesgue measure.
- **c)** Show that every continuous functions are measurable.
- **d)** What is measure theory?
- **e)** Let $A = \bigcap_{i=1}^{\infty} G_i$, where G_i are open sets. Find the complement of A.
- **f)** Show that outer measure is translation invariant.
- **g)** Give an example of a function where strict inequality occurs in Fatou 's Lemma.
- **h)** Show that $\int_1^{\infty} \frac{dx}{x} = \infty$.
- **i)** State Lebesgue's Differentiation Theorem.
- **j)** Show that the Lebesgue set of a function $f \in L(a, b)$ contains any point at which f is continuous.

Part-II

Long Answer Type Questions (Answer Any five)

Q2 a) Show that the outer measure of an interval is its length. **(5+5)**

b) If f is a measurable function and $f = g$ a.e., then show that g is measurable.

Q3 a) Let $\{f_n\}$ be a sequence of continuous functions, $f_n : X \to \mathbb{R}$ and let $f_n \to f$ uniformly; then prove that f is a continuous function. **(5+5)**

b) Prove that every non-empty perfect subset of real number is uncountable.

Q4 a) Show that every countable set has measure zero. **(5+5)**

b) Show that every interval is measurable.

Q5 **a)** Show that if f is a non-negative measurable function, then $f = 0$ a.e. if, and only if, $\int f\, dx = 0$. **(5+5)**

b) Let $\{f_n\}$ be a sequence of non-negative measurable functions. Then show that $\int \sum_{n=1}^{\infty} f_n\, dx = \sum_{n=1}^{\infty} \int f_n\, dx$.

Q6 **a)** State and prove Fatou 's Lemma. **(5+5)**

b) Let f be a bounded function define on the finite interval $[a, b]$, then f is Riemann integrable over $[a, b]$ if, and only if, it is continuous a.e.

Q7 **a)** Show that the derivates of a continuous function are measurable. **(5+5)**

b) Construct a monotone function with a discontinuity at each rational in [0,1].

Q8 **a)** Evaluate at $x = 0$ the four derivates of the continuous function **(5+5)**

$$f(x) = \begin{cases} ax\sin^2\frac{1}{x} + bx\cos^2\frac{1}{x}, & x > 0 \\ 0, & x = 0 \\ cx\sin^2\frac{1}{x} + dx\cos^2\frac{1}{x}, & x < 0 \end{cases}$$ where $a < b, c < d$.

b) Show that $BV[a, b]$ is a vector space over the real numbers.